AIRCRAFT EQUATION OF MOTION

Prepared by

A.Kaviyarasu
Assistant Professor
Department of Aerospace Engineering
Madras Institute Of Technology
Chromepet, Chennai

- In developing the response side of the aircraft force equations, several additional assumptions will be made.
- First an aircraft is assumed to be a rigid body
  - This assumes that the different parts of the aircraft are not moving with respect to each other
- The mass of the aircraft is also assumed to be constant, which is reasonable over a relatively short duration of time.

- This assumption allows Newton's 2nd law to be rewritten as

$$m\left[\frac{d\left(\bar{V}\right)}{dt}\right]_{inertial} = m\bar{a}_{inertial} = \bar{F}$$

- Newton's 2nd law is only valid with respect to an inertial reference frame, the equations can be expressed in the vehicle body axis system.

- If the equations are expressed in the body axis system, the fact that the system is rotating with respect to an inertial reference frame.

- Expressing the body measurement in the inertial reference frame

$$\left(\bar{a}_{Inertial}\right)_{body} = \dot{\bar{V}}_{body} + \bar{\omega}_{body} \times \bar{V}_{body}$$

The velocity vector in the body axis system, $\bar{V}_{body}$*, is defined as*

$$\bar{V}_{body} = U\hat{i} + V\hat{j} + W\hat{k}$$

where U, V, *and* W *are the velocities in the* x, y, *and* z *body axes, respectively*

- The aircraft angular rate in the body axis system, $\bar{\omega}_{body}$ Body, is defined as

$$\bar{\omega}_{body} = P\hat{i} + Q\hat{j} + R\hat{k}$$

$P,\ Q,\ and\ R\ are\ the\ roll,\ pitch,\ and\ yaw\ rates,\ respectively\ expressed\ in\ the\ body\ axis.$

$$\left(\bar{a}_{Inertial}\right)_{body} = \begin{bmatrix} \dot{U} \\ \dot{V} \\ \dot{W} \end{bmatrix}_{body} + \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ P & Q & R \\ U & V & W \end{bmatrix}_{body}$$

$$\left(\bar{a}_{Inertial}\right)_{body} = \begin{bmatrix} \dot{U} + QW - RV \\ \dot{V} + RU - PW \\ \dot{W} + PV - QU \end{bmatrix}_{body}$$

- Multiplying the inertial acceleration in the body axis system by the mass m of the aircraft yields the three force equations

$$m \begin{bmatrix} \dot{U} + QW - RV \\ \dot{V} + RU - PW \\ \dot{W} + PV - QU \end{bmatrix}_{body} = \begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix}_{body} = \bar{F}_{body}$$

$$
\underbrace{
\begin{aligned}
m\left(\dot{U} + QW - RV\right) &= F_x \\
m\left(\dot{V} + RU - PW\right) &= F_y \\
m\left(\dot{W} + PV - QU\right) &= F_z
\end{aligned}
}_{\textit{response side of the force equation}}
$$

APPLIED FORCES

- The previous section developed the left-hand side, or response side, of the force equations. The right-hand side of each equation consists of the applied forces that act on the aircraft. They consist of the gravity forces, the aerodynamic forces, and the thrust forces.

$$m\left(\dot{U} + QW - RV\right) = F_{Gx} + F_{Ax} + F_{Tx}$$

$$m\left(\dot{V} + RU - PW\right) = F_{Gy} + F_{Ay} + F_{Ty}$$

$$m\left(\dot{W} + PV - QU\right) = F_{Gz} + F_{Az} + F_{Tz}$$

- The left-hand sides of the equations (response equation) were developed in the body axis system, the right-hand side (the above equation) must also be in the body axis system.
- Therefore, each of the forces must be represented in the body axis system for the previous equations to be valid.

- The gravity forces, aerodynamic forces, and thrust forces were previously determined in the body axis system. Therefore, the three force equations in the body axis system are

$$\underbrace{\begin{aligned}
m\left(\dot{U}+QW-RV\right) &= -mg\sin\Theta+\left(-D\cos A+L\sin A\right)+T\cos\phi_T \\
m\left(\dot{V}+RU-PW\right) &= mg\sin\phi\cos\Theta+F_{Ay}+F_{Ty} \\
m\left(\dot{W}+PV-QU\right) &= mg\cos\phi\cos\Theta+\left(-D\sin A-L\cos A\right)-T\sin\phi_T
\end{aligned}}_{\textit{Three force equation}}$$

MOMENT EQUATION

- The three moment equations are determined by applying Newton's 2nd law in a manner similar to the three force equations. Newton's 2nd law states that the time rate of change in the angular momentum of the aircraft is equal to the applied moments acting on the aircraft, namely,

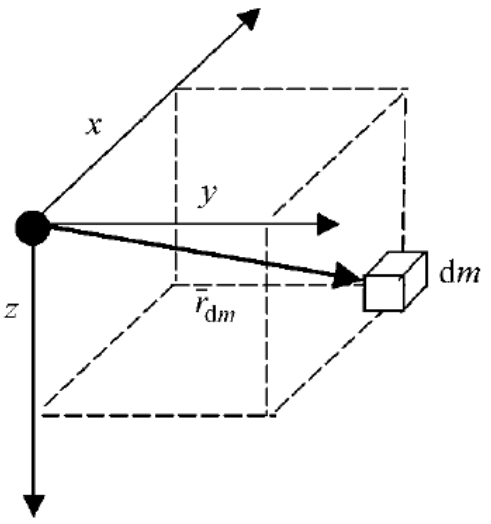
$$\left[\frac{d\bar{H}}{dt}\right]_{Inertial} = \bar{M}$$

- H is the angular momentum of the aircraft and is defined as

$$\bar{H} = \bar{r} \times \left(m\bar{V}\right)$$

Response side of moment equations

- A six-step procedure will be used to methodically build up the response side of the three moment equations. This provides both a mathematical and physical insight into the equations.



Differential mass in body axis system

- Step 1. The first step is to examine a small elemental mass, dm, of the aircraft that is located at some distance from the aircraft's center of gravity. It will be assumed that the elemental mass is rotating about the aircraft center of gravity with a positive roll rate, pitch rate, and yaw rate (P, Q, and R, respectively). The distance from the center of gravity to the small mass is defined as

$$\bar{r}_{dm} = x\hat{i} + y\hat{j} + z\hat{k}$$

where x, y, and z are the distances in the x, y, and z axes of the body axis system

- Step 2. Next an expression is developed for the velocity of the small mass, dm, solely because of its rotation about the center of gravity. The velocity for the movement of the center of gravity of the aircraft was taken into account in the development of the three force equations. The velocity $\bar{V}_{dm}$ of the mass relative to the center of gravity is determined using the expression

$$\bar{V}_{dm} = \left[\frac{d\bar{r}_{dm}}{dt} \right]_{body} + \bar{\omega}_{body} \times \bar{r}_{dm}$$

- Because the aircraft was previously assumed to be a rigid body, $\bar{r}_{dm}$ is constant

$$\left[\frac{d\bar{r}_{dm}}{dt} \right]_{body} = 0$$

$$\bar{V}_{dm} = \bar{\omega}_{body} \times \bar{r}_{dm}$$

$$\bar{V}_{dm} = \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ P & Q & R \\ x & y & z \end{bmatrix}$$

$$\bar{V}_{dm} = (Qz - Ry)\hat{i} + (Rx - Pz)\hat{j} + (Py - Qx)\hat{k}$$

- Step 3. Next an expression is developed for the linear momentum dm of solely because of its rotation about the center of gravity. The linear momentum is found simply by multiplying the mass times the velocity, namely,

$$Linear\ Momentum = dm\bar{V}$$

$$= dm\left[\left(Qz - Ry\right)\hat{i} + \left(Rx - Pz\right)\hat{j} + \left(Py - Qx\right)\hat{k}\right]$$

- Step 4. An expression for the angular momentum of the differential mass, dm, is developed using

$$d\bar{H}_{dm} = \bar{r}_{dm} \times \left(dm\bar{V}_{dm} \right)$$

$$d\bar{H}_{dm} = \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & z \\ dm\left(Qz - Ry\right) & dm\left(Rx - Pz\right) & dm\left(Py - Qx\right) \end{bmatrix}$$

- After carrying out the cross product and regrouping the terms, the three components of the angular momentum are

$$dH_x = P\left(y^2 + z^2\right)dm - Qxy\, dm - Rxz\, dm$$

$$dH_y = Q\left(x^2 + z^2\right)dm - Ryz\, dm - Pxy\, dm$$

$$dH_z = R\left(x^2 + y^2\right)dm - Pxz\, dm - Qyz\, dm$$

- Step 5. The next step is to integrate the expressions for the angular momentum of dm over the entire aircraft. Because P, Q, and R are not functions of the mass, they can be taken outside of the integration. Therefore, the three components for the angular momentum of the entire aircraft are

$$H_x = \int dH_x = P\int \left(y^2 + z^2\right) dm - Q\int xy\ dm - R\int xz\ dm$$
$$H_y = \int dH_y = Q\int \left(x^2 + z^2\right) dm - R\int yz\ dm - P\int xy\ dm$$
$$H_z = \int dH_z = R\int \left(x^2 + y^2\right) dm - P\int xz\ dm - Q\int yz\ dm$$

The moment of inertia are defined as

$$I_{xx} = \int \left(y^2 + z^2 \right) dm$$

$$I_{yy} = \int \left(x^2 + z^2 \right) dm$$

$$I_{zz} = \int \left(x^2 + y^2 \right) dm$$

- The moments of inertia are indications of the resistance to rotation about that axis (i.e, Ixx indicates the resistance to rotation about the x axis of the aircraft).

$$\textit{The product of inertia are} \quad \begin{aligned} I_{xy} &= \int xy\, dm \\ I_{xz} &= \int xz\, dm \\ I_{yz} &= \int yz\, dm \end{aligned}$$

- The products of inertia are an indication of the symmetry of the aircraft. Substituting the moments and products of inertia

$$H_x = PI_{xx} - QI_{xy} - RI_{xz}$$

$$H_y = QI_{yy} - RI_{yz} - PI_{xy}$$

$$H_z = RI_{zz} - PI_{xz} - QI_{yz}$$

$$\bar{H} = H_x\hat{i} + H_y\hat{j} + H_z\hat{k}$$

$$\bar{H} = \bar{I}\,\bar{\omega}$$

- This can also be easily found by applying an expression for angular momentum usually developed in basic physics courses, which is
- Where $\bar{I}$ is the aircraft's inertia tensor and omega is the aircraft's angular rate. The inertia tensor for an aircraft is

$$\bar{I} = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{xy} & I_{yy} & -I_{yz} \\ -I_{xz} & -I_{yz} & I_{zz} \end{bmatrix}_{Body}$$

$$H_B = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{xy} & I_{yy} & -I_{yz} \\ -I_{xz} & -I_{yz} & I_{zz} \end{bmatrix}_{Body} \begin{bmatrix} P \\ Q \\ R \end{bmatrix}$$

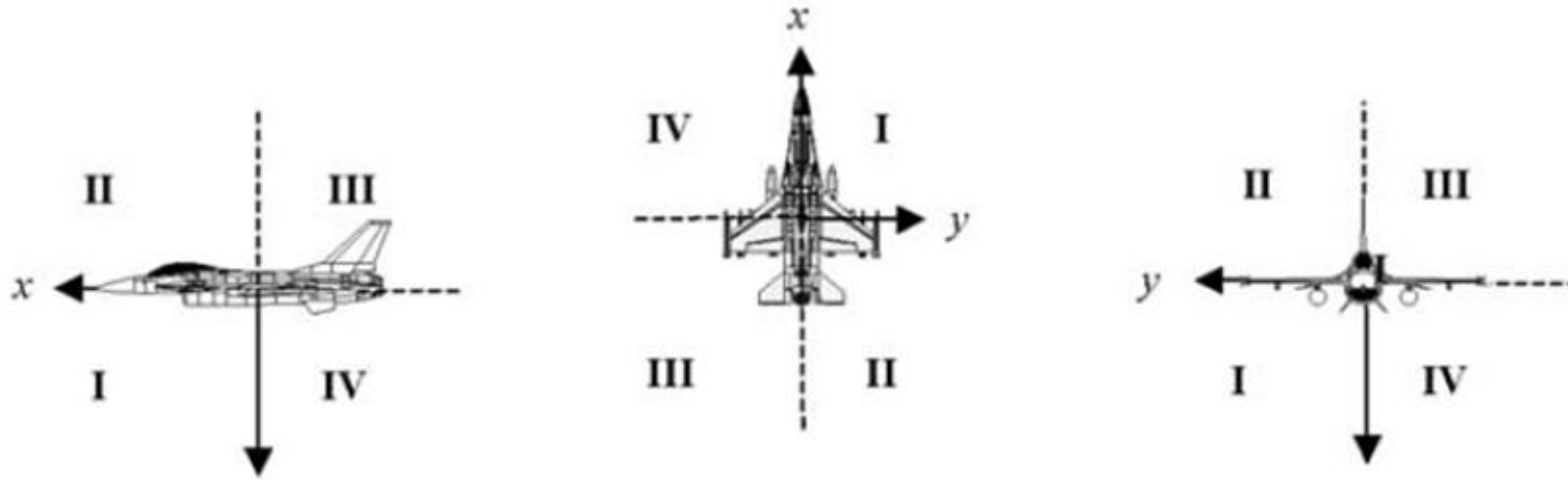
$$H_x = PI_{xx} - QI_{xy} - RI_{xz}$$

$$H_y = QI_{yy} - RI_{yz} - PI_{xy}$$

$$H_z = RI_{zz} - PI_{xz} - QI_{yz}$$

If the aircraft is assumed to have an xz plane of symmetry the I_{xy} and I_{yz} products of inertia are zero.

An aircraft has an xz plane of symmetry when the left side of the aircraft is a mirror image of the right side about the xz plane.



$$I_{xz} = \int xz\, dm \neq 0 \qquad I_{xy} = \int xy\, dm = 0 \qquad I_{yz} = \int yz\, dm = 0$$

Aircraft product of inertia

The I_{xz} *in the first figure not necessarily zero because the aircraft is not symmetrical from top to bottom about the xz plane.*

Notice for I_{xy} *and* I_{yz} *the reflection plane symmetry between quadrants I and IVand II and III.*

This leads to a zero value for both products of inertia. Also notice that we do not have reflection plane symmetry for the case of I_{xz} *therefore, it has a nonzero value.*

The angular momentum components for the aircraft become

$$H_x = PI_{xx} - RI_{xz}$$
$$H_y = QI_{yy}$$
$$H_z = RI_{zz} - PI_{xz}$$

$$\bar{H} = \left(PI_{xx} - RI_{xz}\right)\hat{i} + \left(QI_{yy}\right)\hat{j} + \left(RI_{zz} - PI_{xz}\right)\hat{k}$$

- Step 6. After the angular momentum vector of the aircraft has been determined, the final step is to take the time rate of change of the angular momentum vector with respect to inertial space but represented in the aircraft body axis system. The same relationship used in developing the acceleration with respect to an inertial reference frame from the force equations can be used, namely

$$\left[\frac{d\bar{H}}{dt}\right]_{Inertial} = \left[\frac{d\bar{H}}{dt}\right]_{body} + \omega_{body} \times \bar{H}_{body}$$

$$\left[\frac{d\bar{H}}{dt}\right]_{body} = \begin{bmatrix} \dot{P}I_{xx} - \dot{R}I_{xz} + P\dot{I}_{xx} - R\dot{I}_{xz} \\ \dot{Q}I_{yy} + Q\dot{I}_{yy} \\ \dot{R}I_{zz} - \dot{P}I_{xz} + R\dot{I}_{zz} - P\dot{I}_{xz} \end{bmatrix}_{body}$$

- Assuming that the mass distribution of the aircraft is constant, such as neglecting fuel slosh, the moments and products of inertia do not change with time $i.e\ \dot{I}_{xx}, \dot{I}_{yy}\ and\ \dot{I}_{zz}\ are\ all\ zero$

$$\left[\frac{d\bar{H}}{dt}\right]_{body} = \begin{bmatrix} \dot{P}I_{xx} - \dot{R}I_{xz} \\ \dot{Q}I_{yy} \\ \dot{R}I_{zz} - \dot{P}I_{xz} \end{bmatrix}_{body}$$

$$\omega \times \bar{H}_{body} = \begin{bmatrix} i & j & k \\ P & Q & R \\ PI_{xx} - RI_{xz} & QI_{yy} & RI_{zz} - PI_{xz} \end{bmatrix}_{body}$$

$$\omega \times \bar{H}_{body} = \begin{bmatrix} Q\left(RI_{zz} - PI_{xz}\right) - RQI_{yy} \\ R\left(PI_{xx} - RI_{xz}\right) - P\left(RI_{zz} - PI_{xz}\right) \\ PQI_{yy} - Q\left(PI_{xx} - RI_{xz}\right) \end{bmatrix}_{body}$$

- Grouping terms yields

$$\left[\frac{d\bar{H}}{dt}\right]_{Inertialbody} = \begin{bmatrix} \dot{P}I_{xx} + QR\left(I_{zz} - I_{yy}\right) - \left(\dot{R} + PQ\right)I_{xz} \\ \dot{Q}I_{yy} - PR\left(I_{zz} - I_{xx}\right) + \left(P^2 - R^2\right)I_{xz} \\ \dot{R}I_{zz} + PQ\left(I_{yy} - I_{xx}\right) + \left(QP - \dot{P}\right)I_{xz} \end{bmatrix}_{body}$$

- The above equation yields the three moment equations of motion in the body axis system, where the left-hand side represents the response of the aircraft and the right-hand side consists of the applied moments.

$$\dot{P}I_{xx} + QR\left(I_{zz} - I_{yy}\right) - \left(\dot{R} + PQ\right)I_{xz} = L$$

$$\dot{Q}I_{yy} - PR\left(I_{zz} - I_{xx}\right) + \left(P^2 - R^2\right)I_{xz} = M$$

$$\underbrace{\dot{R}I_{zz}}_{\substack{\text{Angular} \\ \textit{acceleration} \\ \textit{terms}}} + \underbrace{PQ\left(I_{yy} - I_{xx}\right)}_{\substack{\text{Gyro} \\ \textit{precession} \\ \textit{terms}}} + \underbrace{\left(QP - \dot{P}\right)I_{xz}}_{\substack{\text{Coupling} \\ \textit{terms}}} = N$$

- L, M, and N are the rolling moment, pitching moment, and yawing moment, respectively. Unfortunately, the letter L is used to also represent lift.

Thank You